B.Sc. Part-I (Semester-I) Examination

MATHEMATICS

(Differential & Integral Calculus)

Paper—II

Time : Three Hours] [Maximum Marks : 60

Note :—(1) Question No. **1** is compulsory. Attempt once.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternatives (**1** mark each) : 10

(i) The value of $\lim\limits_{x\to 0} \frac{\sin x}{x}$ is :

(a) 0 (b) 1

(c) ∞ (d) None of these

(ii) If $y = e^{-2x}$, then y_{11} is :

(a) $-2^{11} e^{-2x}$ (b) $2^{11} e^{-2x}$

(c) $-2^{11} e^{2x}$ (d) None of these

(iii) The series :

$$x - \frac{x^3}{3!} + \ldots\ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \ldots\ldots$$

is the expansion of function :

(a) sin x (b) sinh x

(c) cos x (d) cosh x

(iv) $|x - x_0| < \delta$ represents :

(a) $x_0 - \delta < x < x_0 + \delta$ (b) $x_0 + \delta < x < x_0 - \delta$

(c) $x_0 - \delta \le x < x_0 + \delta$ (d) $x_0 - \delta < x \le x_0 + \delta$

(v) If f be differentiable on (a, b) and $f'(x) = 0$, $\forall\, x \in [a, b]$, then f(x) is :

(a) Monotonic increasing in [a, b] (b) Monotonic decreasing in [a, b]

(c) Constant in [a, b] (d) None of these

(vi) For $f(x) = x^2$; in [1, 3] then the value of 'C' by Lagrange's mean value theorem is :

(a) $\frac{6}{13}$ (b) 2

(c) 0 (d) 1

(vii) The area bounded by the curve $x = g(y)$; y-axis and $y = a$, $y = b$ is :

(a) $\int_a^b y\,dx$ (b) $\int_a^b x\,dy$

(c) $\int_a^b y^2\,dx$ (d) $\int_a^b x^2\,dy$

(viii) The functions f and g be :

(i) continuous in [a, b]

(ii) derivable in (a, b) and

(iii) $g'(x) \neq 0$ for all $x \in (a, b)$.

These are the hypothesis of mean value theorem by :

(a) Rolle's (b) Lagrange's

(c) Cauchy's (d) Leibnitz

(ix) The function f(x) has the removable discontinuity if :

(a) $f(x^+) \neq f(x^-)$ (b) $f(x^+) = f(x^-) \neq f(x)$

(c) $f(x^+)$, $f(x^-)$ do not exist (d) None of these

(x) $\frac{d}{dx}$ cosh x is :

(a) sinh x (b) –sinh x

(c) h sinh x (d) –h sinh x

UNIT—I

2. (a) If $\lim_{x \to x_0} f(x) = \ell$ and $\lim_{x \to x_0} g(x) = m$, then prove that :

$$\lim_{x \to x_0} [f(x) + g(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$= \ell + m.$$ 4

(b) Prove that the function defined by $f(x) = x^2$ is continuous for all $x \in R$. 3

(c) Using definition of limit, prove that :

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - x - 6}{x - 3} = 14.$$ 3

3. (p) Define limit of a function and show that the limit of a function if it exist, is unique. 1+3

(q) Prove that $\lim_{x \to 2} x^2 = 4$; by using ϵ-δ definition. 3

(r) If $f(x)=\frac{e^{1/x}}{1+e^{1/x}},\ x \neq 0,$

$= 0, \quad , x = 0$

then show that f(x) has a simple discontinuity at x = 0. 3

UNIT—II

4. (a) Prove that if f(x) is differentiable at $x = x_0$, then it is continuous at $x = x_0$. Is converse of this statement true ? Justify. 5

(b) Evaluate :

$$\lim_{x \to 0} (\cos x)^{\cot^2 x}.$$ 3

(c) If y = A sin mx + B cos mx, then prove that $y_2 + m^2 y = 0$. 2

5. (p) If $y = \sin(m \sin^{-1} x)$, then show that :

(i) $(1 - x^2)y_2 - xy_1 + m^2 y = 0$

(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - m^2)y_n = 0$. 5

(q) If $y=\frac{1}{ax+b}$, then prove that $y_n=\frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$. 3

(r) Evaluate :

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}.$$ 2

UNIT—III

6. (a) State and prove Lagrange's mean value theorem. 4

(b) Verify Cauchy mean value theorem for the functions :

$f(x) = e^x$ and $g(x) = e^{-x}$ in [a, b]. 3

(c) Expand sin x in powers of $x-\frac{\pi}{2}$, upto first four terms. 3

7. (p) State and prove Cauchy's mean value theorem. 4

(q) Expand $3x^3 + 4x^2 + 5x - 3$ about the point x = 1 by Taylor's theorem. 3

(r) Verify the Rolle's theorem for the function :

$$f(x)=\frac{\sin x}{e^x} \text{ in } [0, \pi].$$ 3

UNIT—IV

8. (a) If u = f(x, y, z) is a homogeneous function of degree n, then show that :

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$ 4

(b) Verify Euler's theorem for $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$. 3

(c) If $u = e^x (x \cos y - y \sin y)$, then find the value $u_{xx} + u_{yy}$. 3

9. (p) If $u = f(x, y)$ be homogeneous function of degree n then prove that :

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are homogeneous functions of degree 'n – 1' in x, y and

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$. 4

(q) If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, then show that :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$ 3

(r) If $u = \log \frac{x^4 - y^4}{x - y}$, $x \neq y$, then prove that :

(i) $x u_x + y u_y = 3$

(ii) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = -3$. 3

UNIT—V

10. (a) Prove that :

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx.$$ 4

(b) Evaluate :

$$\int \frac{x^2 + 2x + 3}{\sqrt{x^2 + x + 1}} dx.$$ 3

(c) Show that '8a' is the length of an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$; $0 \leq t \leq 2\pi$. 3

11. (p) Prove that :

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

Hence evaluate $\int \tan^3 x \, dx$. 4

(q) Find the area bounded by the x-axis, the curve $y = c \cosh \frac{x}{c}$ and the ordinates $x = 0$, $x = a$. 3

(r) Show that length of the curve $y = \log \sec x$ between the points, where $x = 0$ and $x = \frac{\pi}{3}$ is $\log_e (2 + \sqrt{3})$. 3